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Blow-Up Rate Estimates for Semilinear Parabolic Systems[1]

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This paper deals with the blow-up rate estimates of positive solutions for semi-linear parabolic systems with null Dirichlet boundary conditions. The upper and [illegible] lower bounds.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the estimates of blow-up rate of positive solutions to the semilinear parabolic systems

$$\begin{cases} u_t = \Delta u + v^p, & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + u^q, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where $p, q > 0$, $\Omega \subset \Re^n$ is a bounded domain with smooth boundary $\partial\Omega$, $u_0(x)$ and $v_0(x)$ are continuous and nonnegative functions, and vanish on $\partial\Omega$.

When $pq > 1$, it is well known that for sufficiently "large" initial data $u_0(x)$ and $v_0(x)$, the solution $(u(x,t), v(x,t))$ of (1.1) blows up in finite time, say T. Furthermore, the blow-up of u and v are simultaneous [2].

When $p, q > 1$ and $\Omega = B(0, R)$, the ball of $\Re^n$ centered at the origin of radius R, Caristi and Mitidieri in [1], by using some modifications of the arguments of [7], discussed the blow-up estimates of positive solutions of (1.1). Their results are the following:

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THEOREM A. *Assume that the following hypotheses are satisfied*:

(i) $u_0, v_0\colon \overline{B(0, R)} \to \Re^1$ *are nonnegative, radial, and symmetrically decreasing* C^1 *functions, and vanish on* $\partial B(0, R)$,

(ii) (u, v) *is the classical solution of* (1.1), *defined on* $B(0, R) \times (0, T)$, *where* T *is the maximum existence time, and* $T < +\infty$,

(iii) $\lim_{t \to T} u(0, t) = \lim_{t \to T} v(0, t) = +\infty$,

(iv) $u_t(x, t) \geqslant 0$, $v_t(x, t) \geqslant 0$ *for* $(x, t) \in B(0, R) \times (0, T)$,

(v) $u_t(\cdot, t)$ *and* $v_t(\cdot, t)$ *achieve the maximum at* 0, *for every* $t \in (0, T)$.

If one of the following conditions is satisfied:

(g_1) $n \leqslant 2$ *and* $1 < q \leqslant p$,

(g_2) $n \geqslant 3$ *and* $1 < q \leqslant p$, $1/(p+1) + 1/(q+1) > (n-2)/n$,

(g_3) $n \geqslant 5$ *and* $1 = q < p < (n+4)/(n-4)$,

then

$$u(x, t) \leqslant C(T-t)^{-(p+1)/(pq-1)}, \tag{1.2}$$

$$v(x, t) \leqslant C(T-t)^{-(q+1)/(pq-1)} \tag{1.3}$$

for all $(x, t) \in B(0, R) \times (0, T)$.

The main purposes of this short paper are to improve the above results in the following two aspects. The first one is to relax the conditions on the initial data $(u_0(x), v_0(x))$ and the domain Ω, and delete the restrictions on the dimension n and parameters p, q in Theorem A. By using a simple method to obtain the estimates (1.2) and (1.3), the upper bounds of blow-up rate. The second one is to give the estimates of the lower bounds of blow-up rate. Our results read as follows.

THEOREM 1.1. *Assume that* $p, q \geqslant 1$ *and* $pq > 1$. *If the initial data* u_0 *and* v_0 *are nonnegative and nontrivial* C^1 *functions, and vanish on* $\partial\Omega$, *such that the classical solution* (u, v) *of* (1.1) *blows up in finite time* $T < +\infty$, *and*

$$u_t(x, t) \geqslant 0, \qquad v_t(x, t) \geqslant 0 \qquad \text{for} \quad (x, t) \in \Omega \times (0, T),$$

then there exists a function $C(x)$ *such that*

$$u(x, t) \leqslant C(x)(T-t)^{-(p+1)/(pq-1)} \qquad \text{for all} \quad (x, t) \in \bar{\Omega} \times [0, T), \tag{1.4}$$

$$v(x, t) \leqslant C(x)(T-t)^{-(q+1)/(pq-1)} \qquad \text{for all} \quad (x, t) \in \bar{\Omega} \times [0, T). \tag{1.5}$$

In particular, if $\Omega = B(0, R)$, and u_0, v_0 are radial and symmetrically decreasing, then

$$u(x, t) \leqslant C(T-t)^{-(p+1)/(pq-1)}, \qquad v(x, t) \leqslant C(T-t)^{-(q+1)/(pq-1)} \tag{1.6}$$

for some positive constant C and all $(x, t) \in \Omega \times [0, T)$.

THEOREM 2. *Assume that $pq > 1$, the positive solution (u, v) of (1.1) blows up in finite time T, and $u_t \geqslant 0$, $v_t \geqslant 0$. If there exists $C > 0$ such that (1.6) holds, then there exists $c > 0$ such that*

$$\max_{\bar{\Omega}} u(x, t) \geqslant c(T-t)^{-(p+1)/(pq-1)} \qquad \text{for all} \quad 0 \leqslant t < T, \tag{1.7}$$

$$\max_{\bar{\Omega}} v(x, t) \geqslant c(T-t)^{-(q+1)/(pq-1)} \qquad \text{for all} \quad 0 \leqslant t < T. \tag{1.8}$$

COROLLARY 1. *Under the assumption of Theorem 1 (certainly, for the radial case), there exist c and C such that*

$$c(T-t)^{-(p+1)/(pq-1)} \leqslant \max_{\bar{\Omega}} u(x, t) \leqslant C(T-t)^{-(p+1)/(pq-1)},$$

$$c(T-t)^{-(q+1)/(pq-1)} \leqslant \max_{\bar{\Omega}} v(x, t) \leqslant C(T-t)^{-(q+1)/(pq-1)},$$

for all $0 < t < T$.

Let us remark briefly on our results. If $p = q$, and $u_0 = v_0$, (1.1) reduces to a scalar problem, namely

$$\begin{cases} u_t = \Delta u + u^p, & x \in \Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{1.9}$$

Weissler [7] gave an upper bound for the rate of blow-up of positive solutions to (1.9) under the similar assumptions as that of [1]. Our present results give also better improvements of [7].

2. PROOF OF THEOREMS

Proof of Theorem 1. By the results of [2, 6], $u(x, t)$ and $v(x, t)$ are positive in $\Omega \times (0, T)$. Since $(u(x, t), v(x, t))$ blows up in finite time $T < +\infty$, and $u_t(x, t) \geqslant 0$, $v_t(x, t) \geqslant 0$ for all $(x, t) \in \Omega \times (0, T)$, it follows that for any t_0: $0 < t_0 < T$, $u_t(x, t_0) \not\equiv 0$ or $v_t(x, t_0) \not\equiv 0$ on $\bar{\Omega}$ (otherwise, $(u(x, t_0), v(x, t_0))$ is a positive equilibrium solution of (1.1), and hence

$(u(x,t), v(x,t))$ can not blow up in finite time). Let $w^* = u_t$, $z^* = v_t$, then we have

$$\begin{cases} w_t^* = \triangle w^* + pv^{p-1}z^*, & \\ z_t^* = \triangle z^* + qu^{q-1}w^*, & x \in \Omega, \quad t_0 \leqslant t < T, \\ w^* = z^* = 0, & x \in \partial\Omega, \quad t_0 \leqslant t < T, \\ w^*(x, t_0) \geqslant 0, \qquad z^*(x, t_0) \geqslant 0, & \\ w^*(x, t_0) \not\equiv 0 \quad \text{or} \quad z^*(x, t_0) \not\equiv 0, & x \in \bar{\Omega}. \end{cases}$$

By the maximum principle it follows that

$$w^*(x,t) > 0, \qquad z^*(x,t) > 0 \qquad \text{for all} \quad (x,t) \in \Omega \times (t_0, T),$$

$$\frac{\partial w^*}{\partial \eta} < 0, \qquad \frac{\partial z^*}{\partial \eta} < 0, \qquad \text{for all} \quad (x,t) \in \partial\Omega \times (t_0, T),$$

where η is the outward normal. Since $\frac{\partial u}{\partial \eta} < 0$ and $\frac{\partial v}{\partial \eta} < 0$ for all $(x,t) \in \partial\Omega \times (0, T)$, by the standard method it follows that for any t_1: $t_0 < t_1 < T$, there exists $\varepsilon > 0$ such that

$$w^*(x, t_1) \geqslant \varepsilon v^p(x, t_1), \qquad z^*(x, t_1) \geqslant \varepsilon u^q(x, t_1) \qquad \text{for all} \quad x \in \bar{\Omega},$$

i.e.,

$$\triangle u + v^p \geqslant \varepsilon v^p, \qquad \triangle v + u^q \geqslant \varepsilon u^q, \qquad \text{for} \quad t = t_1 \quad \text{and all } x \in \bar{\Omega}.$$

Without loss of generality we can think that $t_1 = 0$, that is,

$$\triangle u_0 + v_0^p \geqslant \varepsilon v_0^p, \qquad \triangle v_0 + u_0^q \geqslant \varepsilon u_0^q \qquad \text{for all} \quad x \in \bar{\Omega}.$$

Let $w = u_t - \varepsilon v^p$, $z = v_t - \varepsilon u^q$. We will use the method of [3] to prove $w(x,t) \geqslant 0$, $z(x,t) \geqslant 0$. Indeed,

$$\begin{aligned} w_t - \Delta w &= (u_t - \Delta u)_t - \varepsilon p v^{p-1} v_t + \varepsilon p v^{p-1} \Delta v + \varepsilon p(p-1) v^{p-2} |\nabla v|^2 \\ &\geqslant p v^{p-1} v_t - \varepsilon p v^{p-1} u^q \\ &= p v^{p-1} z, \qquad x \in \Omega, \quad 0 < t < T, \end{aligned}$$

$$z_t - \Delta z \geqslant q u^{q-1} w, \qquad x \in \Omega, \quad 0 < t < T,$$

$$w(x, 0) = \Delta u_0 + v_0^p - \varepsilon v_0^p \geqslant 0, \qquad z(x, 0) \geqslant 0, \quad x \in \Omega,$$

$$w(x,t) = z(x,t) = 0 \qquad \text{for} \quad (x,t) \in \partial\Omega \times [0, T).$$

The maximum principle implies $w \geqslant 0$, $z \geqslant 0$. Therefore,

$$u_t(x,t) \geqslant \varepsilon v^p(x,t), \qquad v_t(x,t) \geqslant \varepsilon u^q(x,t), \qquad x \in \Omega, \quad 0 < t < T. \tag{2.1}$$

For proving (1.4), we can assume that $\lim_{t \to T} u(x, t) = +\infty$ (otherwise there is nothing to prove). By (2.1) we have

$$(uv)_t \geqslant \varepsilon(v^{p+1} + u^{q+1}), \tag{2.2}$$

and $\lim_{t \to T}(uv)(x, t) = +\infty$. For simplicity we denote $\alpha = (1+p)/(pq-1)$, $\beta = (1+q)/(pq-1)$, and let $k = (1+p+q+pq)/(2+p+q)$. Then

$$\frac{k}{p+1} + \frac{k}{q+1} = 1, \qquad \text{and} \qquad \frac{1}{k-1} = \alpha + \beta.$$

Using the Young Inequality in (2.2) yields

$$(uv)_t \geqslant C(uv)^k, \qquad 0 \leqslant t < T.$$

Since $pq > 1$, we have $k > 1$. Integrating the above inequality from t to T and using $\lim_{t \to T}(uv)(x, t) = +\infty$ we have

$$(uv)(x, t) \leqslant C(T-t)^{-1/(k-1)} = C(T-t)^{-(\alpha+\beta)}, \qquad 0 \leqslant t < T. \tag{2.3}$$

Now we prove (1.4) by contradiction. Assume that there exist sequences $\{t_n\}$ and $\{c_n\}$ with $t_n \to T^-$ and $c_n \to +\infty$ as $n \to +\infty$ such that

$$u(x, t_n) \geqslant c_n(T-t_n)^{-(p+1)/(pq-1)} = c_n(T-t_n)^{-\alpha}.$$

Then one has by (2.1)

$$\begin{aligned}
v(x, t) &\geqslant v(x, t_n) + \varepsilon \int_{t_n}^{t} u^q(x, \tau)\, d\tau \\
&\geqslant \varepsilon u^q(x, t_n)(t-t_n) \\
&\geqslant \varepsilon c_n^q(T-t_n)^{-\alpha q}\,(t-t_n), \qquad && t_n \leqslant t < T, \\
u(x, t) &\geqslant u(x, t_n) + \varepsilon \int_{t_n}^{t} v^p(x, \tau)\, d\tau \\
&\geqslant \varepsilon^{1+p} c_n^{pq}(T-t_n)^{-\alpha pq} \int_{t_n}^{t} (\tau - t_n)^p\, d\tau \\
&= \frac{1}{p+1}\, \varepsilon^{1+p} c_n^{pq}(T-t_n)^{-\alpha pq}\,(t-t_n)^{p+1}, \qquad && t_n \leqslant t < T.
\end{aligned}$$

Thus we get

$$(uv)(x, t) \geqslant \frac{1}{p+1}\, \varepsilon^{2+p} c_n^{q(1+p)}(T-t_n)^{-\alpha q(1+p)}\,(t-t_n)^{p+2}, \qquad t_n \leqslant t < T.$$

Taking $t = z_n \triangleq (T + t_n)/2$ in the above inequality, one has

$$\begin{aligned}(uv)(x, z_n) &\geqslant \frac{1}{p+1}\, \varepsilon^{2+p} c_n^{q(1+p)} (T - t_n)^{-\alpha q(1+p)}\, (1/2)^{p+2}\, (T - t_n)^{p+2} \\ &= \frac{1}{p+1}\, (\varepsilon/2)^{p+2}\, c_n^{q(1+p)} (T - t_n)^{-(\alpha+\beta)} \\ &= \frac{1}{p+1}\, (\varepsilon/2)^{p+2}\, c_n^{q(1+p)} 2^{-(\alpha+\beta)} (T - z_n)^{-(\alpha+\beta)}.\end{aligned}$$

We get a contradiction with (2.3) because $c_n \to +\infty$ as $n \to +\infty$. Hence there exists $C(x) > 0$ such that (1.4) holds. The proof of (1.5) is similar.

Finally, when $\Omega = B(0, R)$, and u_0, v_0 are radial and symmetrically decreasing, then for any given $t > 0$, so do $u(\cdot, t)$ and $v(\cdot, t)$ by the standard method. In view of (1.4) and (1.5) one has

$$\begin{aligned} u(x, t) &\leqslant u(0, t) \leqslant C(0)(T - t)^{-(p+1)/(pq-1)}, \\ v(x, t) &\leqslant v(0, t) \leqslant C(0)(T - t)^{-(q+1)/(pq-1)} \end{aligned}$$

for all $(x, t) \in \bar{\Omega} \times [0, t)$. The proof is completed. ∎

Proof of Theorem 2. We will use the integral equations of $u(x, t)$ and $v(x, t)$ to complete the proof. Let $\Gamma(x, t)$ be the fundamental solution for the heat equation, namely,

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp(-|x|^2/(4t)).$$

Then for $0 < z < t < T$ and $x \in \Omega$ we have Green's identity(see [4, 5]),

$$\begin{aligned} u(x, t) = &\int_\Omega \Gamma(x - y, t - z)\, u(y, z)\, dy + \int_z^t \int_\Omega \Gamma(x - y, t - \tau)\, v^p(y, \tau)\, dy\, d\tau \\ &+ \int_z^t \int_{\partial\Omega} \Gamma(x - y, t - \tau) \frac{\partial u(y, \tau)}{\partial \eta}\, dS_y\, d\tau, \\ v(x, t) = &\int_\Omega \Gamma(x - y, t - z)\, v(y, z)\, dy + \int_z^t \int_\Omega \Gamma(x - y, t - \tau)\, u^q(y, \tau)\, dy\, d\tau \\ &+ \int_z^t \int_{\partial\Omega} \Gamma(x - y, t - \tau) \frac{\partial v(y, \tau)}{\partial \eta}\, dS_y\, d\tau, \end{aligned}$$

where η is the exterior normal vector on $\partial\Omega$. Denote $F(t) = \max_{\bar{\Omega}} u(x, t)$, $G(t) = \max_{\bar{\Omega}} v(x, t)$. Then $F(t)$ and $G(t)$ are non-decreasing. Since $\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} \leqslant 0$ on $\partial\Omega$, we have

$$F(t) \leqslant F(z) + \int_z^t G^p(\tau)\, d\tau \leqslant F(z) + G^p(t)(T-z), \qquad 0<z<t<T,$$

$$G(t) \leqslant G(z) + \int_z^t F^q(\tau)\, d\tau \leqslant G(z) + F^q(t)(T-z), \qquad 0<z<t<T. \quad (2.4)$$

Consequently,

$$\begin{aligned} F(t) &\leqslant F(z) + (T-z)(G(z) + F^q(t)(T-z))^p \\ &\leqslant F(z) + 2^p(T-z)\, G^p(z) + 2^p(T-z)^{p+1}\, F^{pq}(t), \qquad 0<z<t<T, \end{aligned} \quad (2.5)$$

$$G(t) \leqslant G(z) + 2^q(T-z)\, F^q(z) + 2^q(T-z)^{q+1}\, G^{pq}(t), \qquad 0<z<t<T. \quad (2.6)$$

Because $F(t) \to +\infty$ as $t \to T$, it follows that for any z: $0<z<T$, there exists t: $z<t<T$ such that $F(t) = 2[F(z) + 2^p(T-z)\, G^p(z)]$. Thus, by (2.5) we have

$$\begin{aligned} F(z) + 2^p(T-z)\, G^p(z) &\leqslant 2^p(T-z)^{p+1} F^{pq}(t) \\ &= 2^{p(q+1)}(T-z)^{p+1}\, [F(z) + 2^p(T-z)\, G^p(z)]^{pq}, \end{aligned}$$

$$F(z) + 2^p(T-z)\, G^p(z) \geqslant 2^{-p(q+1)/(pq-1)}(T-z)^{-(p+1)/(pq-1)}, \qquad 0<z<T.$$

Similarly, if we choose t such that $z<t<T$ and $G(t) = 2[G(z) + 2^q(T-z)\, F^q(z)]$, it follows from (2.6) that

$$G(z) + 2^q(T-z)\, F^q(z) \geqslant 2^{-q(p+1)/(pq-1)}(T-z)^{-(q+1)/(pq-1)}, \qquad 0<z<T. \quad (2.7)$$

We will prove (1.7) by contradiction. Assume that there exist sequences $\{t_n\}$ and $\{\varepsilon_n\}$ with $t_n \to T^-$ and $\varepsilon_n \to 0^+$ as $n \to +\infty$ such that

$$F(t_n) \leqslant \varepsilon_n (T-t_n)^{-(p+1)/(pq-1)}. \quad (2.8)$$

Then it follows from (2.7) that

$$\begin{aligned} G(t_n) &\geqslant 2^{-q(p+1)/(pq-1)}(T-t_n)^{-(q+1)/(pq-1)} \\ &\quad - 2^q(T-t_n)\, \varepsilon_n^q (T-t_n)^{-q(p+1)/(pq-1)} \\ &= 2^{-q(p+1)/(pq-1)}(T-t_n)^{-(q+1)/(pq-1)} - (2\varepsilon_n)^q\, (T-t_n)^{-(q+1)/(pq-1)} \\ &= (2^{-q(p+1)/(pq-1)} - 2^q \varepsilon_n^q)(T-t_n)^{-(q+1)/(pq-1)}. \end{aligned} \quad (2.9)$$

Choosing $a>1$ is so large that

$$Ca^{-(q+1)/(pq-1)} < 2^{-q(p+1)/(pq-1)}, \tag{2.10}$$

where C is given by (1.6). Then we have by the second inequality of (1.6)

$$G(T-a(T-t_n)) \leqslant Ca^{-(q+1)/(pq-1)}(T-t_n)^{-(q+1)/(pq-1)} \tag{2.11}$$

for the large n (where $T-a(T-t_n)>0$ as n is large). Since $a>1$, we have $T-a(T-t_n)<t_n$. Taking $t=t_n$, $z=T-a(T-t_n)$ in (2.4), and combining the results with (2.8) and (2.11), we get

$$\begin{aligned} G(t_n) &\leqslant G(T-a(T-t_n)) + F^q(t_n)\, a(T-t_n) \\ &\leqslant Ca^{-(q+1)/(pq-1)}(T-t_n)^{-(q+1)/(pq-1)} + a\varepsilon_n^q(T-t_n)^{-(q+1)/(pq-1)} \\ &= (Ca^{-(q+1)/(pq-1)} + a\varepsilon_n^q)(T-t_n)^{-(q+1)/(pq-1)}. \end{aligned}$$

This inequality combined with (2.9) yields

$$Ca^{-(q+1)/(pq-1)} + a\varepsilon_n^q \geqslant 2^{-q(p+1)/(pq-1)} - 2^q\varepsilon_n^q.$$

This is a contradiction with (2.10) because $\varepsilon_n \to 0$ as $n \to +\infty$. Therefore, (1.7) holds. The proof of (1.8) is similar. The proof is completed. ▮

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